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Automatic nonlinear MPC approximation with closed-loop guarantees

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Proposed framework [1]

User-defined bound ϵ → **Robust MPC**
- Closed-loop guarantees for disturbances on control input bounded by ϵ

Samples of MPC → **ALKIA-X**
- Automatic computation of approximate MPC
- Uniform approximation error bounded by ϵ (Theorem 1)

Approximating function → **Approximate MPC**
- Fast-to-evaluate controller
- Closed-loop guarantees (Corollary 1)

We consider a robust model predictive control (MPC) formulation, which is designed such that the desired closed-loop guarantees remain valid until input disturbances below a user-chosen error bound $\epsilon > 0$. By approximating the feedback law implicitly defined by this MPC up to a tolerance ϵ, the approximate MPC preserves all control-theoretic guarantees induced by the MPC. Hence, approximating the MPC can be cast as a function approximation problem by sampling state and corresponding optimal inputs obtained by solving the MPC offline. To address this function approximation problem, we propose ALKIA-X, the **A**daptive and **L**ocalized **K**ernel **I**nterpolation **A**lgorithm with e**X**trapolated reproducing kernel Hilbert space (RKHS) norm. ALKIA-X automatically computes an explicit function that approximates the MPC with a uniform approximation error ϵ, resulting in a cheap-to-evaluate approximate MPC with guarantees on stability and constraint satisfaction.

Introduction

- Nonlinear MPC is computationally expensive but we require fast evaluation
- Explicit MPC approaches: Mostly for linear MPC or without closed-loop guarantees

Related work: Nonlinear MPC approximation with closed-loop guarantees [2]

- Construct MPC that is robust w.r.t. input disturbances bounded by ϵ
- Sample MPC law $f\colon \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ and learn approximation $h\colon \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ offline with NNs
- Closed-loop guarantees on approximate MPC h if:

$$|f(x) - h(x)| \leq \epsilon \quad \forall x \in \mathcal{X} \tag{1}$$

- Problem: Guaranteeing (1) for NNs is nontrivial and iterative

Solving (1) using kernel interpolation

- Assume f is in reproducing kernel Hilbert space (RKHS) of kernel k
- Kernel interpolation yields error bounds with RKHS norm $\|f\|_k$ and power function P_X [3]:

$$|f(x) - h(x)| \leq P_X(x)\|f\|_k \quad \forall x \in \mathcal{X} \tag{2}$$

ALKIA-X

Properties

(P1). Fast-to-evaluate approximating function;
(P2). Guaranteed satisfaction of (1);
(P3). Bound on worst-case number of required samples;
(P4). Automatic and non-iterative algorithm with well-conditioned computations.

Tools

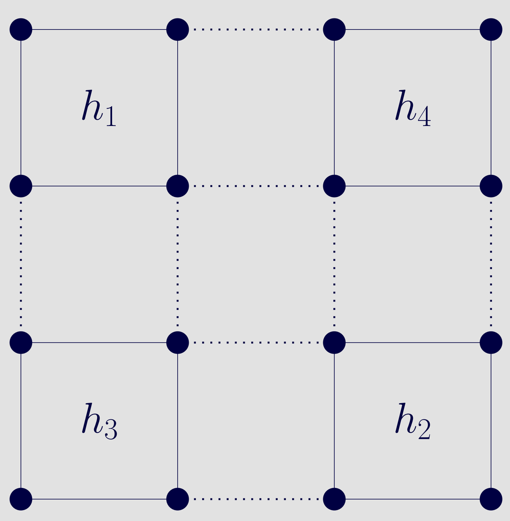
(T1). Localized kernel interpolation;
(T2). Adaptive sub-domain partitioning;
(T3). RKHS norm extrapolation.

Localized kernel interpolation (T1)

- Kernels methods have scalability issues with more samples
- Approach based on local cubes: Localized approximating function that only uses samples within corresponding cube
- Piecewise-defined approximating function

⇓ Online tree search and local kernel evaluation ⇓ Fast-to-evaluate approximating function (P1)

⇓ Guaranteed satisfaction of error bound (P2)

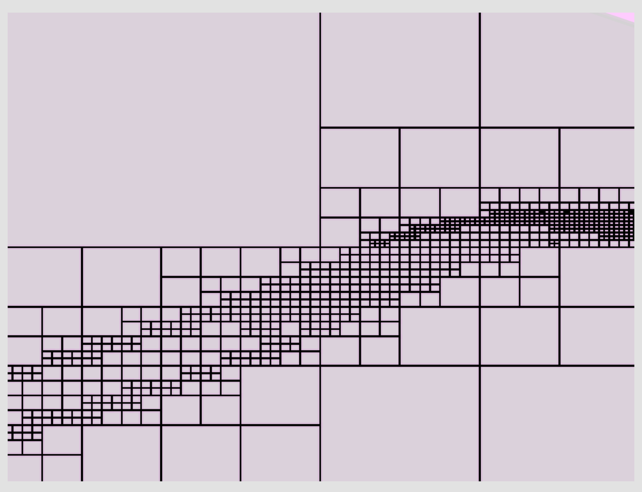


Adaptive sub-domain partitioning (T2)

- Introduce maximum number of samples per domain
- More samples required → partition to sub-domains and reduce length scale accordingly
- Localized kernel interpolation on each sub-domain

⇓ Sample complexity bounds (P3)

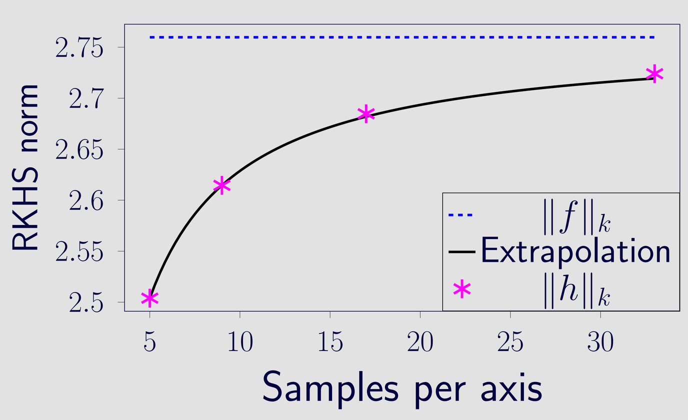
⇓ Well-conditioned computations (P4)



RKHS norm extrapolation (T3)

- We require RKHS norm $\|f\|_k$ to bound (2)
- Oracle assumption on RKHS norm is restrictive
- Exponential function to extrapolate $\|h\|_k$
- Heuristic: Limit upper-bounds $\|f\|_k$

⇓ Automatic and implementable algorithm (P4)



Theorem 1: ALKIA-X

Suppose:
- *Regularity assumptions on kernel k*
- *RKHS norm extrapolation heuristic holds*

Then:
- *ALKIA-X terminates after finite number of samples*
- *Resulting approximation h satisfies (1), i.e., approximation error is uniformly bounded by ϵ*

Further suppose k is the SE-kernel, then:
- *Offline computational complexity: $\mathcal{O}\left(\left(\frac{\sqrt{n}\|f\|_k}{\epsilon}\right)^n\right)$*
- *Online computational complexity $\mathcal{O}\left(\log_2\left(\frac{\sqrt{n}\|f\|_k}{\epsilon}\right)\right)$*

Corollary 1: Closed-loop guarantees for approximate MPC

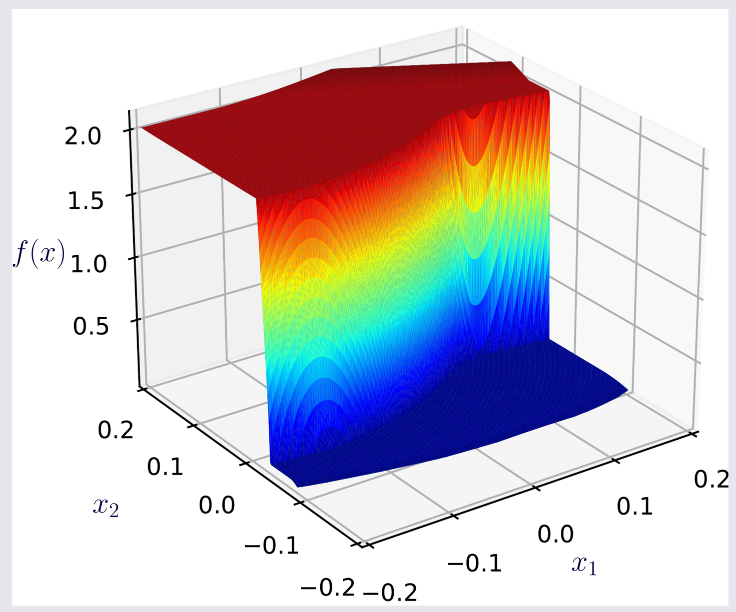
Suppose:
- *Hypotheses of Theorem 1 hold*
- *MPC is robust w.r.t. input disturbances bounded by ϵ*

Then, ALKIA-X yields an approximate controller that inherits closed-loop properties from the MPC, i.e., it ensures:

(i). *Recursive feasibility;*
(ii). *Constraint satisfaction;*
(iii). *Practical asymptotic stability.*

Continuous stirred tank reactor

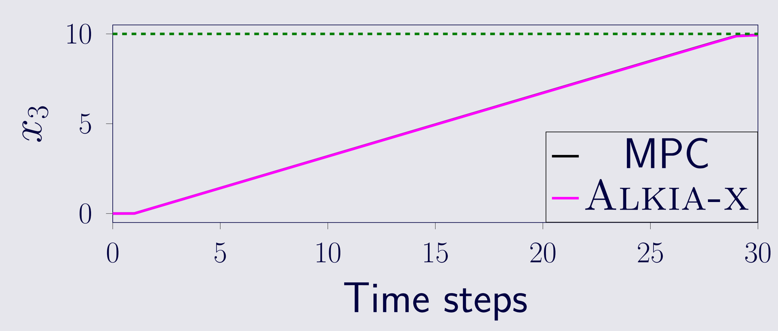
	ALKIA-X	NN [2]
t_{online}	$44.02\,\mu s \pm 0.14\,\mu s$	$3000\,\mu s$
t_{offline}	$10.3\,h$	$500\,h$
card(X)	$1.56 \cdot 10^6$	$1.6 \cdot 10^6$



- Approximating 2-state-1-input nonlinear MPC using ALKIA-X vs. using NNs [2]
- ALKIA-X has 100-times faster online evaluation and 50-times faster offline approximation

Cold atmospheric plasma

	ALKIA-X	MPC (IPOPT)
t_{online}	$96\,\mu s \pm 4\,\mu s$	$507\,\text{ms} \pm 6\,\text{ms}$
t_{offline}	$69\,h$	[-]
card(X)	$8.22 \cdot 10^5$	[-]



- Approximating 3-state-2-input nonlinear MPC using ALKIA-X subject to memory constraints
- Approximate MPC with ALKIA-X almost identical trajectory as MPC
- ALKIA-X has 5000-times faster online evaluation

Conclusion

- ALKIA-X automatically approximates nonlinear MPC schemes with closed-loop guarantees
- ALKIA-X can automatically approximate wide range of black-box functions with guarantees
- Paper, code, extended abstract, poster: **QR code**

References

[1] A. Tokmak, C. Fiedler, M. N. Zeilinger, S. Trimpe, and J. Köhler. "Automatic nonlinear MPC approximation with closed-loop guarantees". In: *TAC (submitted)* (2023).

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